NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 1 2017

Name: ________________ **Mathematics Class:** 12MZ__

Student Number: _______________

Time Allowed: **55 minutes + 2 minutes reading time**

Available Marks: **37**

Instructions:

- Questions are not of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1-4	5	6	Total
Graphs	/2	/11	/3	/16
Polynomials	/2	/6	/13	/21
				/37

Section I

4 marks
Attempt Questions 1- 4
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 4

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1 Consider the graph of the function $y = \dfrac{x^2}{x+2}$.

What is the equation of the non-vertical asymptote of this function?

(A) $y = 1$

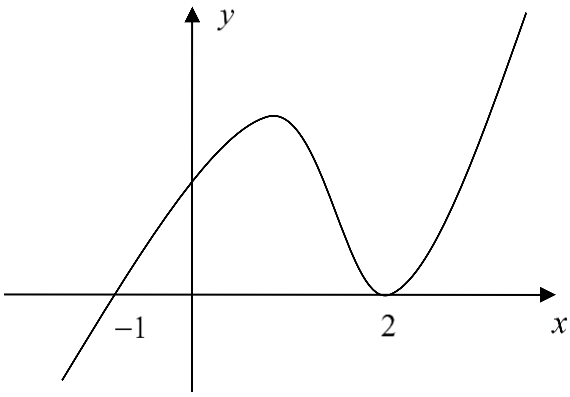
(B) $y = x$

(C) $y = x - 2$

(D) $y = x + 2$

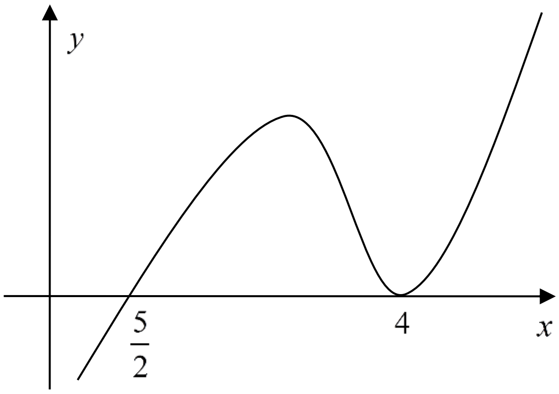
Question 2 on page 3

2 The graph of the function $y = f(x)$ is shown:

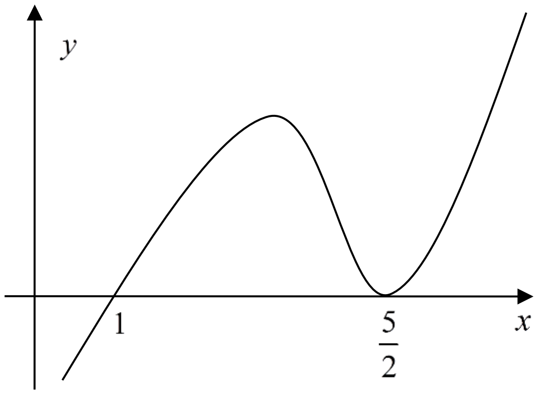


Which of the graphs below (not drawn to scale) could be the graph of $y = f(2x-3)$?

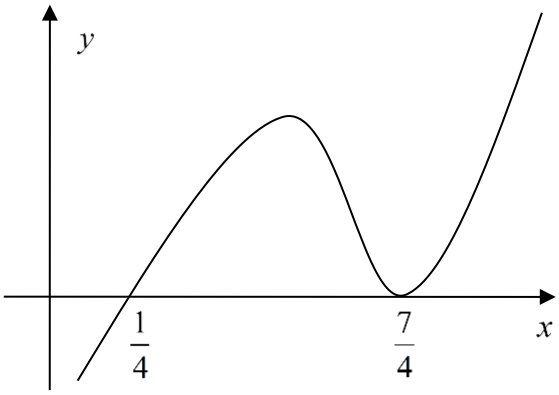
(A)



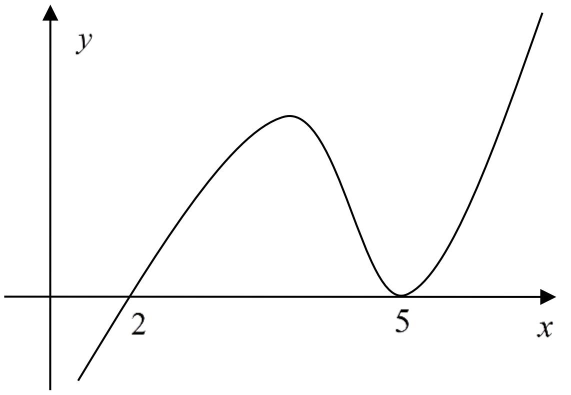
(B)



(C)



(D)



3 α is a complex number and is a double zero of the polynomial $x^3 + bx^2 + cx + d = 0$.

Which of the following statements **must** be true?

(A) $\overline{\alpha}$ is also a zero of the polynomial

(B) the polynomial has a quadratic factor with real coefficients

(C) $3\alpha^2 + 2b\alpha = -c$

(D) all of the above

4 The polynomial equation $ax^n + bx + c = 0$ has roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$.

What is the value of $\displaystyle\sum_{i=1}^{n} \alpha_i^n$?

(A) $\dfrac{nc}{a}$

(B) $-\dfrac{nc}{a}$

(C) $\dfrac{b^2}{a^2} - \dfrac{nc}{a}$

(D) $\dfrac{nc}{a} - \dfrac{b^2}{a^2}$

End of Section I

Section II

33 marks
Attempt Questions 5 - 6
Allow about 48 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 5 – 6, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (Use a SEPARATE writing booklet) **17 marks**

(a) The polynomial $P(x) = x^4 + bx^3 + cx^2 - 24x + 36$ has a double zero at $x = 2$.

(i) Show that $b = -1$ and $c = 1$. **3**

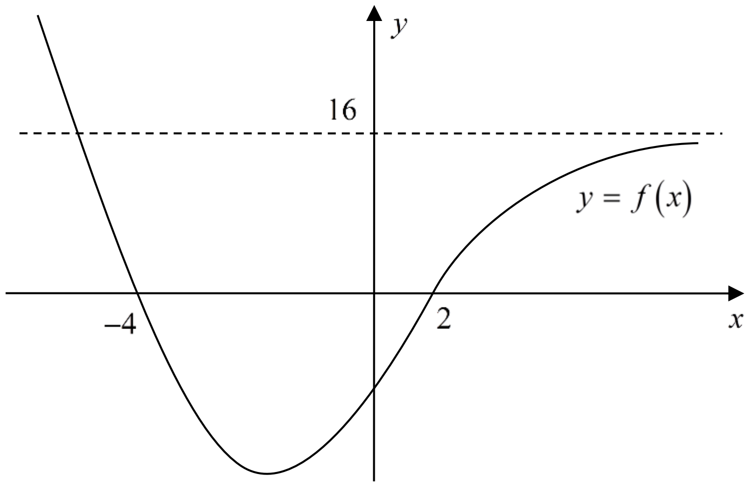
(ii) Hence factorise $P(x)$ fully over the complex numbers. **3**

(b) Use implicit differentiation to find the gradient of the tangent to the curve $(x + y)^3 = -x$ at the point $(1, -2)$. **3**

Question 5 continues on page 6

Answer part (c) on the supplied Graph Answer Sheet

(c) Following is the graph of the function $y = f(x)$.



On the number planes provided, sketch the following graphs, showing all important features:

(i) $y = -|f(x)|$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \log_2 f(x)$ **2**

(iv) $y = f(x^2)$ **2**

End of Question 5

Question 6 (Use a SEPARATE writing booklet) **16 marks**

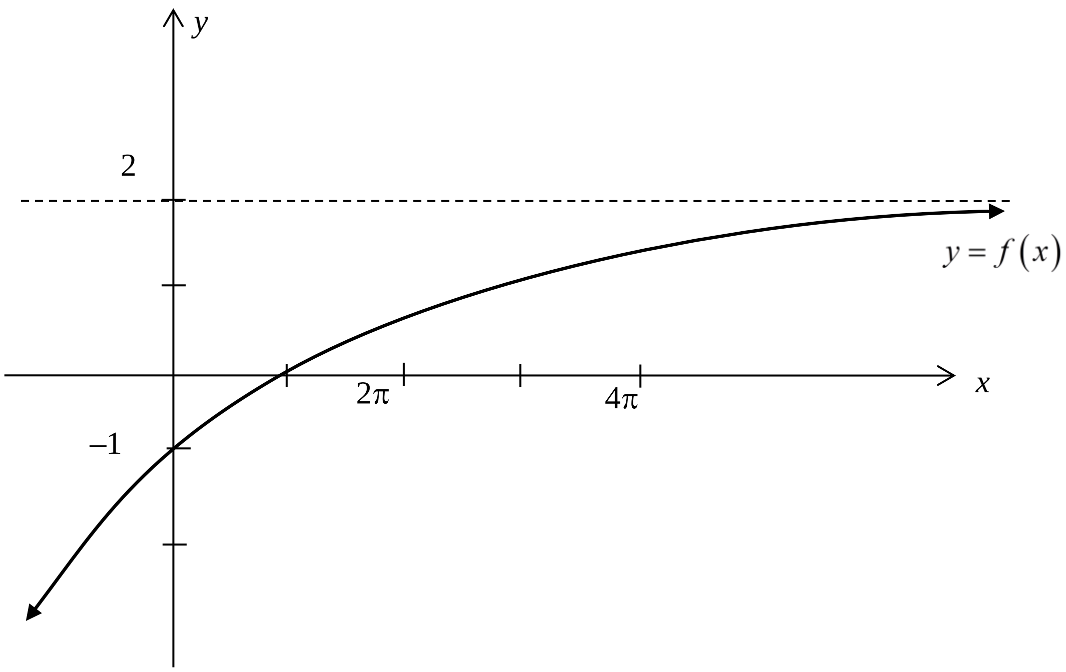
(a) The roots of the equation $x^3 + px + q = 0$ are α, β and γ.

(i) Find the cubic polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Hence or otherwise explain why the equation $x^3 + px + q = 0$, where p and q are real, cannot have three real roots if $p > 0$. **1**

Answer part (b) on the supplied Graph Answer Sheet

(b) Following is the graph of $y = f(x)$. The graph has an x-intercept at $x = \pi$ and $y = 2$ is a horizontal asymptote as shown.



(i) On the number plane provided, use a pencil to sketch the graph of $y = \sin x$ for $0 \le x \le 4\pi$. **1**

(ii) Now using pen, sketch the graph of $y = \sin x \cdot f(x)$ on the same number plane for $0 \le x \le 4\pi$. **2**

Question 6 continues on page 8

Question 6 (continued)

(c) (i) Write down the roots of the equation $z^9 = 1$ in modulus-argument form. **1**

(ii) Hence find the roots of the equation $z^6 + z^3 + 1 = 0$ in modulus-argument form. **2**

(iii) Hence factorise $z^6 + z^3 + 1$ into real quadratic factors. **2**

(iv) Hence find the value of $\left(1+\cos\frac{\pi}{9}\right)\left(1-\cos\frac{2\pi}{9}\right)\left(1-\cos\frac{4\pi}{9}\right)$. **2**

(d) ω is a non-real root of the polynomial equation $x^4 + bx^3 + cx^2 + bx + 1 = 0$, where b and c are real.

(i) Show that $\frac{1}{\omega}$ is also a root of this equation. **1**

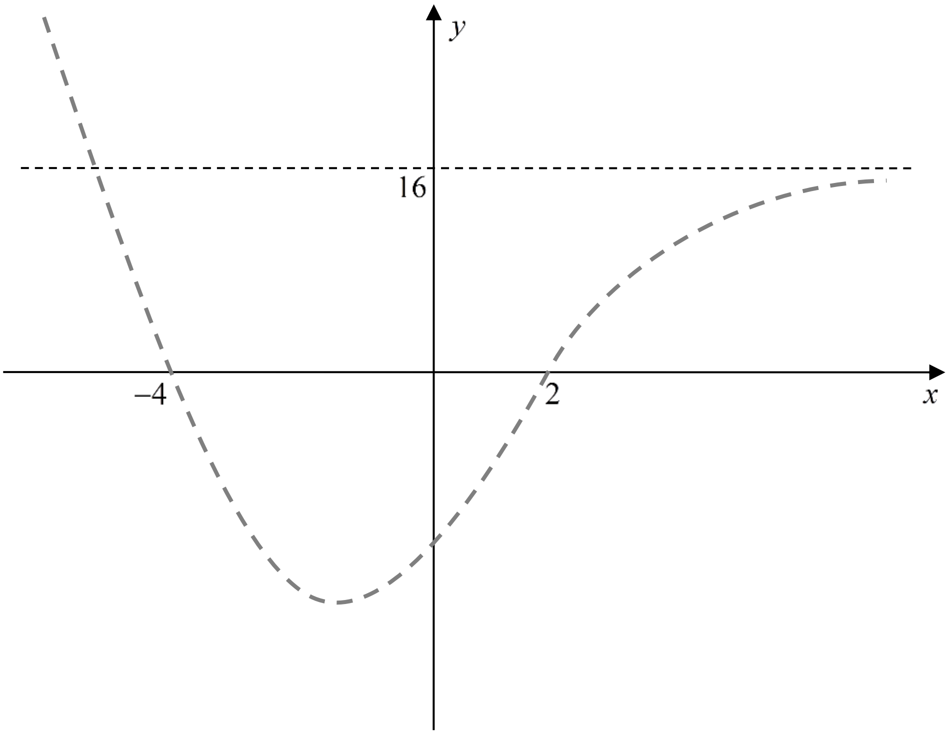
(ii) Show that $|b| > 2|\operatorname{Re}\omega|$. **2**

End of test

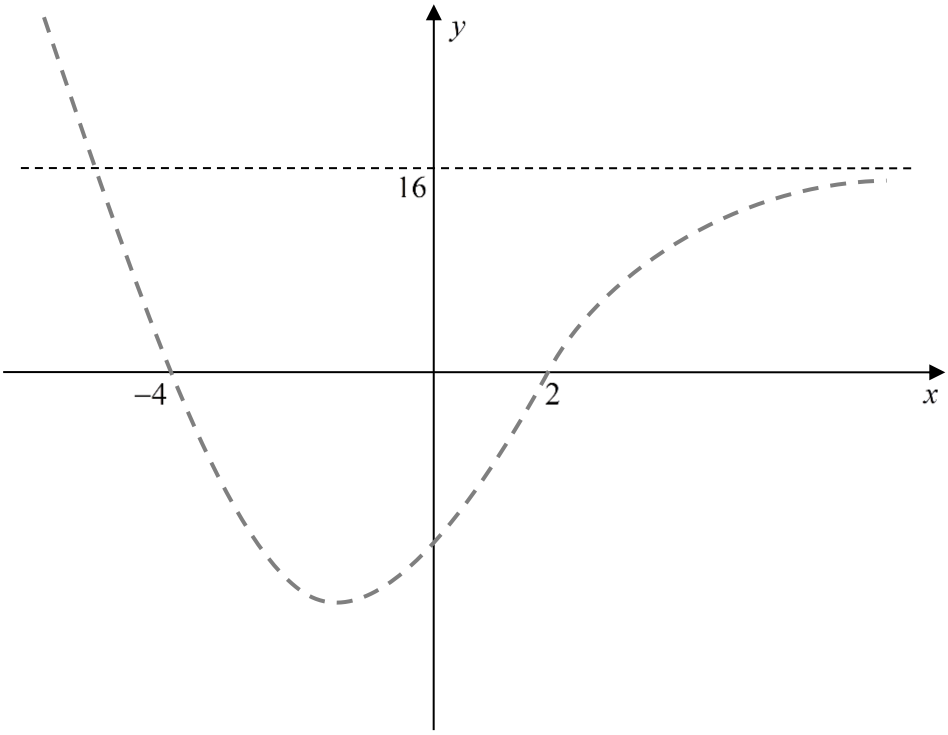
Graph Answer Sheet

Answer question 5 part (c) on this sheet

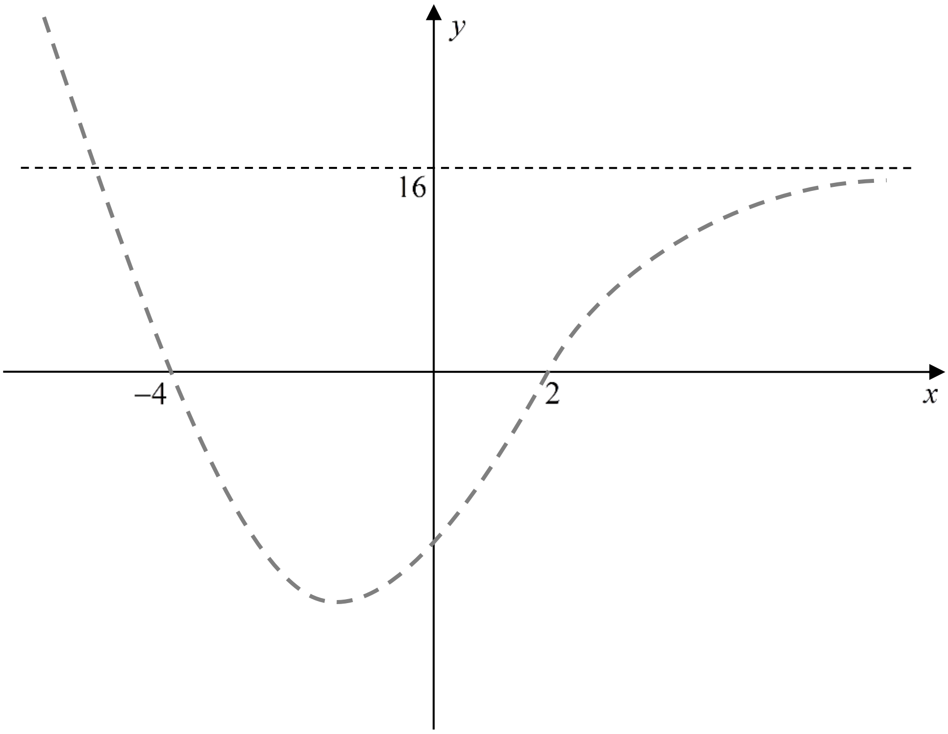
5. (b) (i) $y = -|f(x)|$



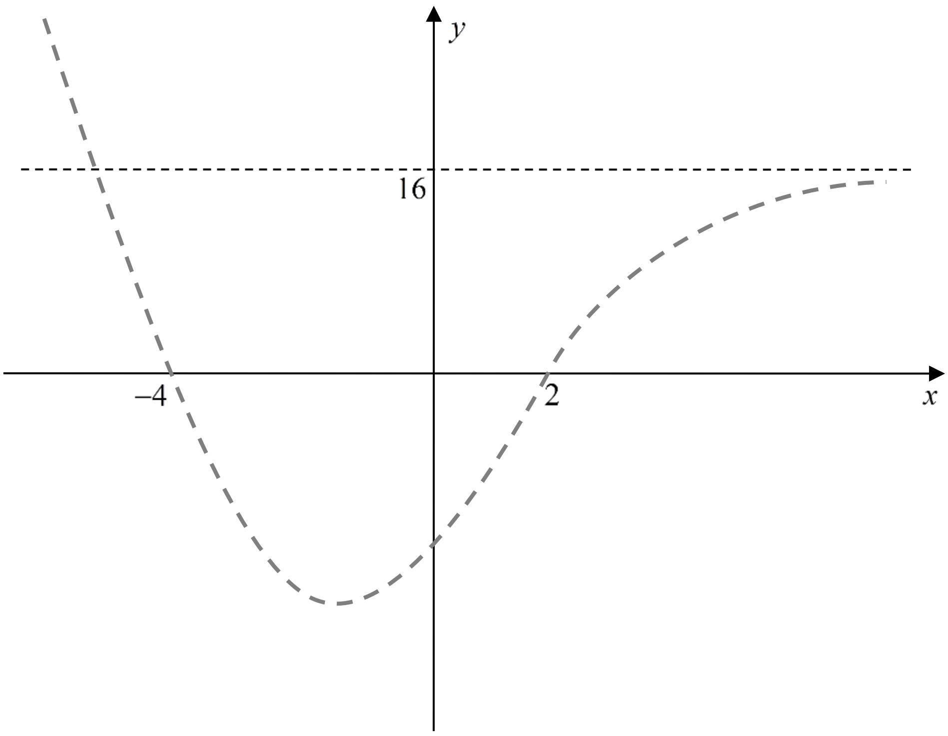
(ii) $y^2 = f(x)$



(iii) $y = \log_2 f(x)$



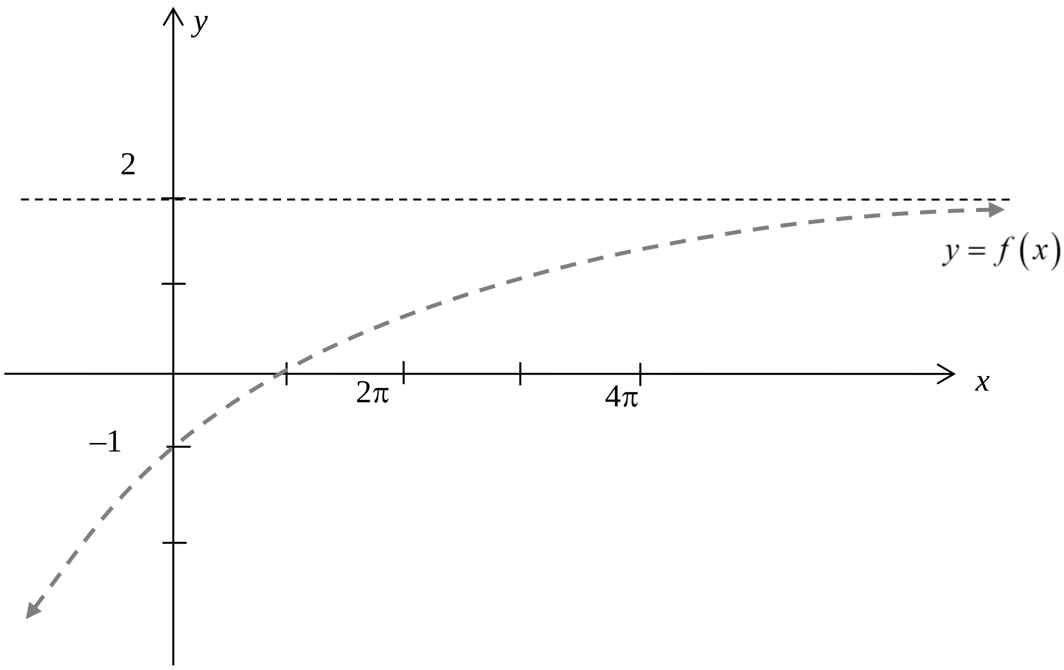
(iv) $y = f(x^2)$



Answer question 6 part (b) on this sheet

6. (b) (i) In pencil, sketch the graph of $y = \sin x$ for $0 \leq x \leq 4\pi$. **1**

(ii) Now using pen, sketch the graph of $y = \sin x \cdot f(x)$ for $0 \leq x \leq 4\pi$. **2**



Ext 2 Solutions

Section I

1. $$\frac{x^2}{x+2} = \frac{x(x+2)-2(x+2)+4}{x+2}$$
$$= x-2+\frac{4}{x+2}$$
As $x \to \infty$, $y \to x-2$ (C)

2. Inverse of $2x-3$ is $\frac{x+3}{2}$.

Applying this function to $x=-1$ and $x=2$ gives 1 and $\frac{5}{2}$. (B)

3. (A) not necessarily true since polynomial is not necessarily real
(B) is not necessarily true for the same reason
Hence clearly not (D) (C)

4. Since $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are all roots:

$$a\alpha_i^n + b\alpha_i + c = 0 \qquad \text{for all } 1 \le i \le n$$

Summing: $$\sum_{i=1}^{n} a\alpha_i^n + \sum_{i=1}^{n} b\alpha_i + \sum_{i=1}^{n} c = 0$$
$$a \cdot \sum_{i=1}^{n} \alpha_i^n = -b \cdot \sum_{i=1}^{n} \alpha_i - nc$$
$$= -b(0) - nc$$
$$\sum_{i=1}^{n} \alpha_i^n = -\frac{nc}{a}$$
(B)

Question 5

(a) The polynomial $P(x) = x^4 + bx^3 + cx^2 - 24x + 36$ has a double zero at $x = 2$.

(i) Show that $b=-1$ and $c=1$. **3**

(i) As $x = 2$ is a double root, then $P(x) = P'(x) = 0$.

$$P(2) = 0$$
$$16 + 8b + 4c - 48 + 36 = 0$$
$$2b + c = -1 \quad ①$$

$$P'(x) = 4x^3 + 3bx^2 + 2cx - 24$$
$$P'(2) = 0$$
$$32 + 12b + 4c - 24 = 0$$
$$3b + c = -2 \quad ②$$

Solving simultaneously: $b=-1$ and $c=1$.

(ii) Hence factorise $P(x)$ fully over the complex numbers. **3**

(ii) $P(x)=x^4-x^3+x^2-24x+36$

Let roots be $2,\ 2,\ \alpha,\ \beta$

Sum of roots: $4+\alpha+\beta=1$
$\alpha+\beta=-3$

Product of roots: $4\alpha\beta=36$
$\alpha\beta=9$

Quadratic expression with these zeros:
$$\begin{aligned}x^2+3x+9 &= x^2+3x+\frac{9}{4}+\frac{27}{4}\\ &=\left(x+\frac{3}{2}\right)^2-\frac{27}{4}i^2\\ &=\left(x+\frac{3}{2}-\frac{3\sqrt{3}}{2}i\right)\left(x+\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)\end{aligned}$$

$$\therefore\ P(x)=(x-2)^2\left[x-\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)\right]\left[x-\left(-\frac{3}{2}-\frac{3\sqrt{3}}{2}i\right)\right]$$

Generally well done.

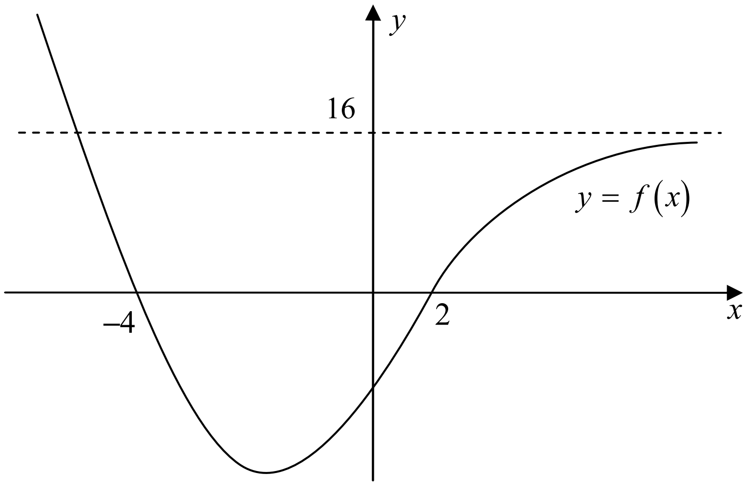
(b) Use implicit differentiation to find the gradient of the tangent to the curve $(x+y)^3=-x$ at the point $(1,\ -2)$. **3**

$$\begin{aligned}(x+y)^3 &= -x\\ 3(x+y)^2\left(1+\frac{dy}{dx}\right) &= -1\\ 1+\frac{dy}{dx} &= -\frac{1}{3(x+y)^2}\\ \frac{dy}{dx} &= -\frac{1}{3(x+y)^2}-1\end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(1,-2)}=-\frac{1}{3(1-2)^2}-1=-\frac{4}{3}$$

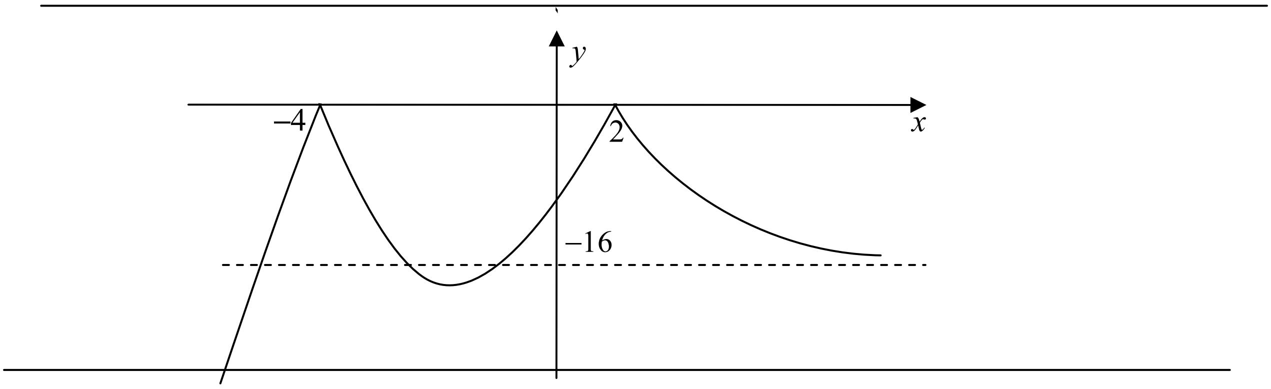
Again generally well done. Many students expanded $(x+y)^3$ to differentiate instead of using the chain rule when differentiating implicitly. And students who did use the chain rule then expanded $3(x+y)^2\left(1+\frac{dy}{dx}\right)$. It is easier to work in factored form.

(c) Following is the graph of the function $y = f(x)$.



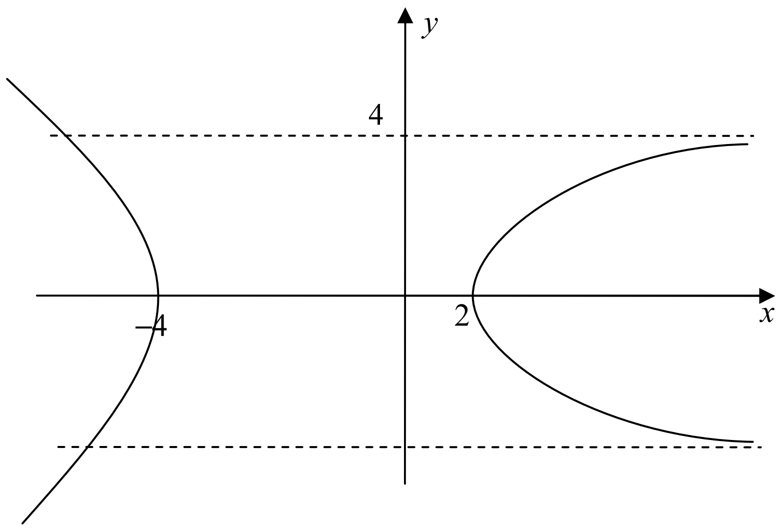
On the number planes provided, sketch the following graphs, showing all important features:

(i) $y = -|f(x)|$ **2**



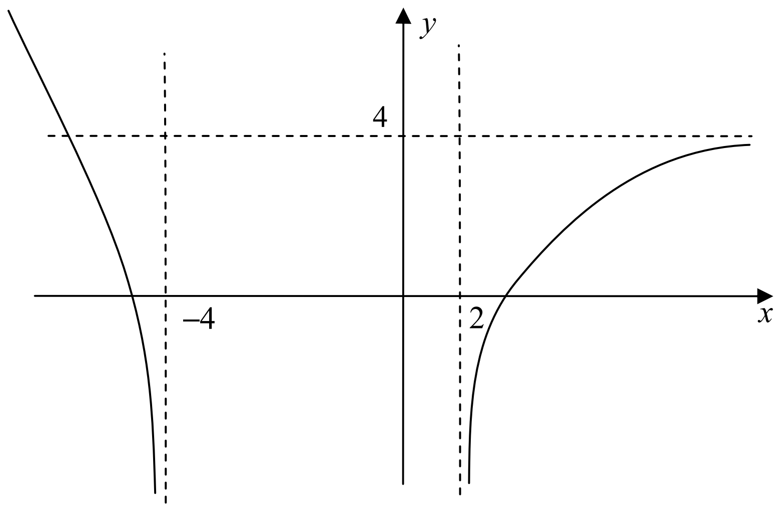
Generally well done. A small number of students got missed up with other transformations or failed to see the negative sign in front of the $|f(x)|$.

(ii) $y^2 = f(x)$ **2**



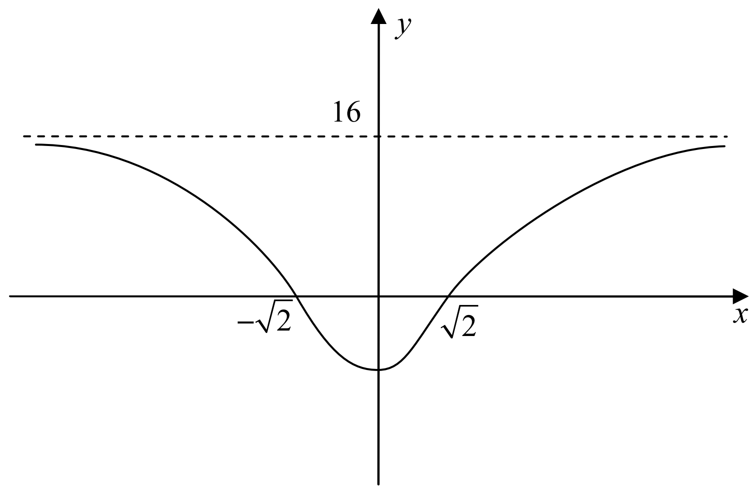
Scale used by students here was very poor generally although this was not penalised. Some students only drew $y = \sqrt{f(x)}$ instead of $y = \pm\sqrt{f(x)}$. Many students are unsure of the relative size of $\sqrt{f(x)}$ in relation to the size of $f(x)$ and some have it the wrong way around! A small number of students drew $y = (f(x))^2$ instead.

(iii) $y = \log_2 f(x)$ **2**



Several students are confused about what to do when $f(x) = 0$. This is where $\log_2(f(x))$ will have an asymptote (not a discontinuity shown by an open circle)

(iv) $y = f(x^2)$ **2**



Several students failed to realise that $y = f(x^2)$ is an even function and drew only one half of the graph required. Intercepts were easy to find and needed to be shown. In this question, the graph of $y = f(x^2)$ lies below the graph of $y = f(x)$ between 0 and 1. Many students did not consider this at all but this was not penalised. Many students also did not realise that the new graph will also have a turning point at $x = 0$ as $y' = f'(x^2).2x$ which is zero at $x = 0$. Again this was not penalised.

6. (a) The roots of the equation $x^3 + px + q = 0$ are α, β and γ.

(i) Find the cubic polynomial equation with roots α^2, β^2 and γ^2. **2**

$$\left(\sqrt{x}\right)^3 + p\sqrt{x} + q = 0$$
$$\sqrt{x}\left(x+p\right) = -q$$
$$x\left(x+p\right)^2 = q^2$$
$$x^3 + 2px^2 + p^2x - q^2 = 0$$

Well done. Most efficient method was to transform the roots as modelled in solution.

(ii) Hence or otherwise explain why the equation $x^3 + px + q = 0$, where p and q are real, cannot have three real roots if $p > 0$. **1**

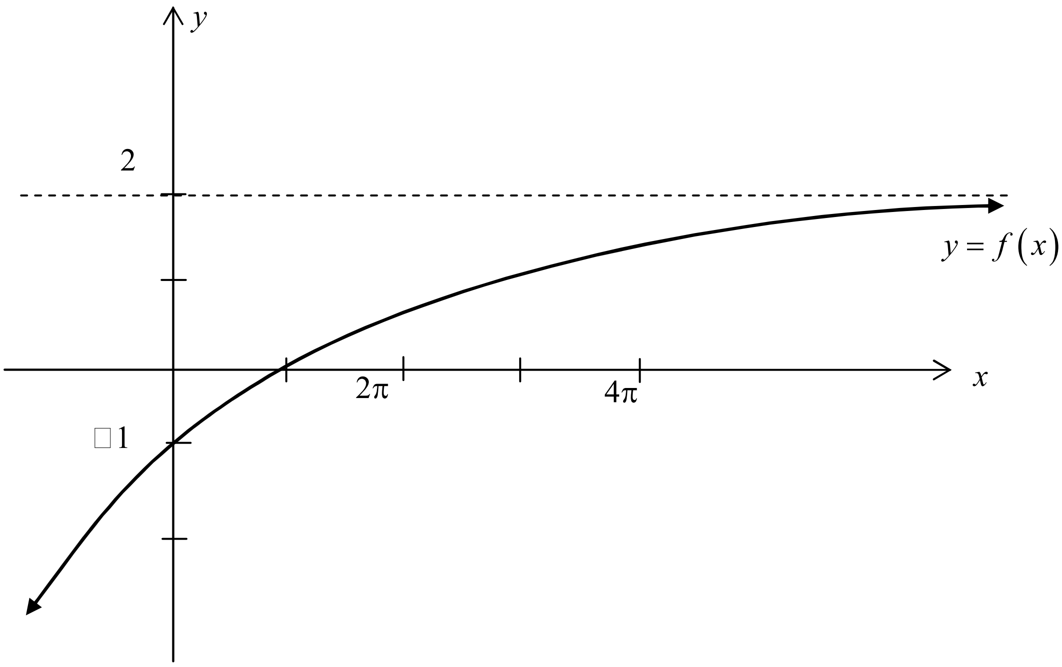
Sum of roots of new equation: $\alpha^2 + \beta^2 + \gamma^2 = -2p$

$< 0 \quad (\text{since } p > 0)$

Since $a^2 \geq 0, \ a \in \mathbb{R}$ then $\alpha^2, \beta^2, \gamma^2$ cannot all be real and hence α, β, γ are not all real.

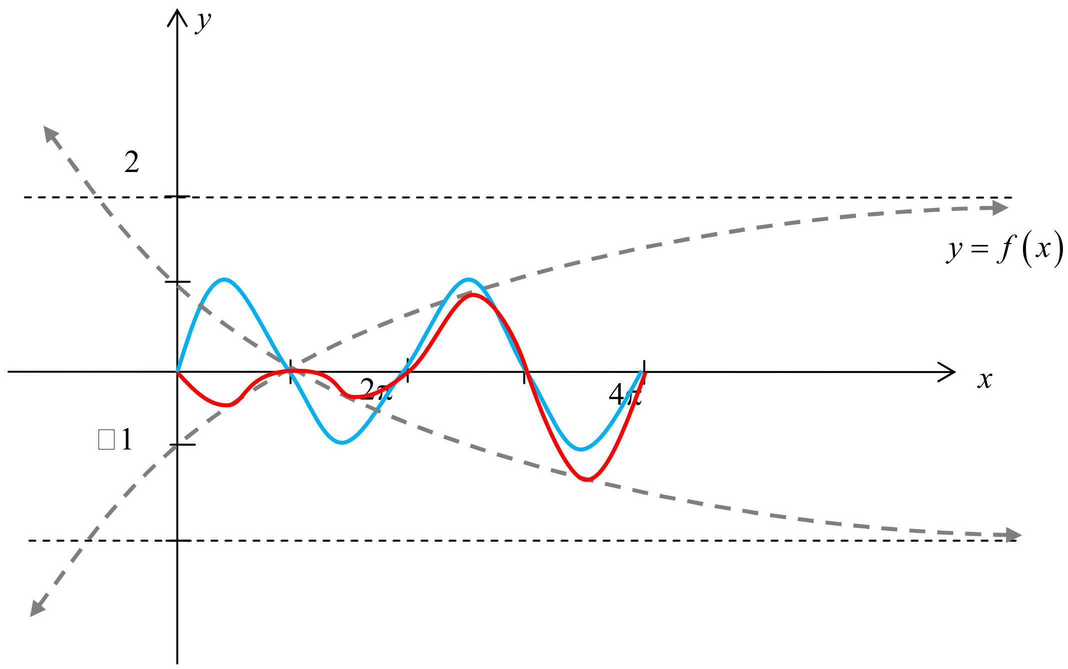
An 'explain why' question requires exactly that, an explanation, generally with words. Many students still need to work on their communication skills in this type of question.

(b) Following is the graph of $y = f\left(x\right)$. The graph has an x-intercept at $x = \pi$ and $y = 2$ is a horizontal asymptote as shown.



(i) On the number plane provided, use a pencil to sketch the graph of $y = \sin x$ for $0 \leq x \leq 4\pi$. **(blue)** **1**

(ii) Now using pen, sketch the graph of $y = \sin x \cdot f\left(x\right)$ for $0 \leq x \leq 4\pi$. **(red)** **2**



(i) **Mostly well done although a small number of students sketched $y = \sin\left(\frac{x}{2}\right)$ instead and managed to lose an easy mark**

(ii) **Overall very well attempted. Some minor teaching points. As $-1 \le \sin x \le 1$, then $-f(x) \le \sin x . f(x) \le f(x)$. So the graphs of $y = f(x)$ and the $y = -f(x)$ act like an envelope within which the new graph oscillates. Most students did not draw the envelope which made it hard to tell whether the students were correctly considering the size of the new graph but penalties applied where transgressions were obvious.. Students are encouraged to lightly draw in the envelope when sketching $y = \sin x . f(x)$ or $y = \cos x . f(x)$ to help keep the magnitude of the transformed graph in the right range. It was also necessary to consider the sign of the product. If both $\sin x$ and $f(x)$ have the same sign the product is positive and otherwise it is negative. Errors in sign incurred a penalty.**

Also most students do not realise that the turning points on the new graph do not line up with the turning points of $\sin x$ and took pains to line them up. Consider $h(x) = f(x).g(x)$. $h'(x) = f'(x)g(x) + f(x)g'(x)$ using the product rule. So $h'(x)$ does not have to be zero where $f'(x) = 0$. This was not penalised.

(c) (i) Write down the roots of the equation $z^9 = 1$ in modulus-argument form. **1**

$$z = \text{cis}\frac{2k\pi}{9}, \quad z = -4, -3, -2, -1, 0, 1, 2, 3, 4$$

Notice the wording in this question. "Write down' means exactly that. Many students wasted time providing a lengthy response to this question. Nearly all students gained this mark.

(ii) Hence find the roots of the equation $z^6 + z^3 + 1 = 0$ in modulus-argument form. **2**

Given $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$, the roots of the given equation are the roots of $z^9 = 1$ less the roots of $z^3 = 1$ $\left[1,\ \text{cis}\left(\pm\frac{2\pi}{3}\right)\right]$.

ie. $\text{cis}\left(\pm\frac{2\pi}{9}\right),\ \text{cis}\left(\pm\frac{4\pi}{9}\right),\ \text{cis}\left(\pm\frac{8\pi}{9}\right)$

The 'hence' in this question means students needed to link their response to part i in order to be awarded full marks. Expressing the roots in cis form using the principal argument is preferred.

(iii) Hence factorise $z^6 + z^3 + 1$ into real quadratic factors. **2**

Forming quadratic whose roots are $\text{cis}\left(\pm\frac{2\pi}{9}\right)$:

$$z^2 - \left[\text{cis}\frac{2\pi}{9} + \text{cis}\left(-\frac{2\pi}{9}\right)\right]z + \text{cis}\frac{2\pi}{9}\cdot\text{cis}\left(-\frac{2\pi}{9}\right) = z^2 - 2\,\text{Re}\left[\text{cis}\frac{2\pi}{9}\right]z + \left|\text{cis}\frac{2\pi}{9}\right|^2$$
$$= z^2 - 2\cos\frac{2\pi}{9}z + 1$$

Doing the same with other conjugate roots:

$$z^6 + z^3 + 1 = \left(z^2 - 2\cos\frac{2\pi}{9}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{9}z + 1\right)\left(z^2 - 2\cos\frac{8\pi}{9}z + 1\right)$$

Again the 'hence' means that working is required to be shown in this question and so a bald answer could only earn one mark. It was obvious that a number of students knew what the answer should be but could not demonstrate how they arrived at the solution.

(iv) Hence find the value of $\left(1+\cos\frac{\pi}{9}\right)\left(1-\cos\frac{2\pi}{9}\right)\left(1-\cos\frac{4\pi}{9}\right)$. **2**

Substituting $z = 1$:

$$3 = \left(2 - 2\cos\frac{2\pi}{9}\right)\left(2 - 2\cos\frac{4\pi}{9}\right)\left(2 - 2\cos\frac{8\pi}{9}\right)$$
$$3 = 2\left(1 - \cos\frac{2\pi}{9}\right)\cdot 2\left(1 - \cos\frac{4\pi}{9}\right)\cdot 2\left(1 - \cos\frac{8\pi}{9}\right)$$
$$\frac{3}{8} = \left(1 - \cos\frac{2\pi}{9}\right)\left(1 - \cos\frac{4\pi}{9}\right)\left(1 + \cos\frac{\pi}{9}\right) \qquad [\text{using } \cos\theta = -\cos(\pi - \theta)]$$

Not enough students realised that the best approach in the question is to substitute a value (in this case z=1). Of those who did correctly substitute there were a small number who factorised out a 2 from the RHS rather than an 8.

A student who at least stated that $\cos\frac{\pi}{9} = -\cos\frac{8\pi}{9}$ earned half a mark.

A number of students tried to expand first in this question, but were generally not successful in arriving at the correct solution. This method was successful if students looked at their answer in part iii, $z^6 + z^3 + 1 = (z^2 - 2\cos\frac{2\pi}{9}z + 1)(z^2 - 2\cos\frac{4\pi}{9}z + 1)(z^2 - 2\cos\frac{8\pi}{9}z + 1)$ and equated the coefficients of the LHS and RHS to obtain the required values.

(d) ω is a non-real root of the polynomial equation $x^4 + bx^3 + cx^2 + bx + 1 = 0$, where b and c are real.

(i) Show that $\frac{1}{\omega}$ is also a root of this equation. **1**

Let $P(x) = x^4 + bx^3 + cx^2 + bx + 1$

$$\begin{aligned} P\left(\frac{1}{\omega}\right) &= \left(\frac{1}{\omega}\right)^4 + b\left(\frac{1}{\omega}\right)^3 + c\left(\frac{1}{\omega}\right)^2 + b\left(\frac{1}{\omega}\right) + 1 \\ &= \frac{1}{\omega^4} + \frac{b}{\omega^3} + \frac{c}{\omega^2} + \frac{b}{\omega} + 1 \\ &= \frac{1 + b\omega + c\omega^2 + b\omega^3 + \omega^4}{\omega^4} \\ &= \frac{0}{\omega^4} \quad \left[\text{since } \omega \text{ is a zero of } P(x)\right] \\ &= 0 \end{aligned}$$

$\therefore \frac{1}{\omega}$ is also a root

There are a number of students who are encouraged to work on their communication skills when answering this type of question. A number of students immediately wrote out $P\left(\frac{1}{\omega}\right)$ and set this expression equal to zero even though this is really what you wanted to show. Too many students wrote out $P\left(\frac{1}{\omega}\right) = \left(\frac{1}{\omega}\right)^4 + b\left(\frac{1}{\omega}\right)^3 + \ldots\ldots$ and then incorrectly just multiplied the RHS of this by ω^4

(ii) Show that $|b| > 2|\operatorname{Re}\omega|$. **2**

Since ω is non-real, $\dfrac{1}{\omega}$ is also non-real.

Since coefficients are real, $\overline{\omega}$ and $\overline{\left(\dfrac{1}{\omega}\right)} = \dfrac{1}{\overline{\omega}}$ are also roots.

Sum of roots: $\omega + \overline{\omega} + \dfrac{1}{\omega} + \dfrac{1}{\overline{\omega}} = -b$

$$\omega + \overline{\omega} + \frac{\omega + \overline{\omega}}{\omega\overline{\omega}} = -b$$

$$2\operatorname{Re}\omega + \frac{2\operatorname{Re}\omega}{|\omega|^2} = -b$$

$$2\operatorname{Re}\omega \cdot \left(1 + \frac{1}{|\omega|^2}\right) = -b$$

$$|b| = 2|\operatorname{Re}\omega|\left(1 + \frac{1}{|\omega|^2}\right)$$

$$> 2|\operatorname{Re}\omega|$$

In this question and part 6di above, a number of students incorrectly assumed that $\dfrac{1}{\omega} = \overline{\omega}$. This is only the case if you know that $|\omega| = 1$. As a result, students didn't realise they could express all four roots as $\omega, \overline{\omega}, \dfrac{1}{\omega}, \dfrac{1}{\overline{\omega}}$ since the coefficients of the polynomial are real. There are a number of possible approaches to this question although the time constraints of the exam meant that it proved difficult to obtain full marks on this question.